

A New Bell Inequality for Two Spin-1 Particle System

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Abstract

For a two spin-1 particles system, we derive a new Bell's type inequality for local hidden variables model. For the singlet state for two spin-1 particles, we show that the inequality is violated while it is satisfied for the direct product state.

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In 1965, Bell demonstrated that an interpretation of quantum theory in terms of local hidden-variables (LHV) is impossible[1], using inequalities now universally known as Bell inequalities. In 1969, Clauser *et al.* [2] showed that these inequalities might be test experimentally. Since then, a series of experiments, of increasing precision and making use of various atomic sources and detection arrangements, have been carried out to test one version or another of Bell inequalities [3,4]. These inequalities studied are based on a $2 \otimes 2$ Hilbert space. For a particle, the case that its dimension is more than 2 has been discussed by the famous Bell-KS theorem[5], which concerns the results of a (counterfactual) set of measurements on quantum state described by a vector in a three dimensional Hilbert space. They consider, for example, measurements of the squares of the three angular momentum components of a spin-1 state. They assume that the corresponding operators commute and can be measured simultaneously, providing one "yes" and two "no's" to the questions "Does the spin component along $\hat{a}, \hat{b}, \hat{a} \times \hat{b}$ vanish?" for any $\hat{a} \perp \hat{b} \in S^2$, the unit sphere in \mathbf{R}^3 . Specker [6] and Bell [7] observed that Gleason's theorem [8] implies that there can be no assignment of "yes's" and "no's" to the vector of S^2 consistent with this requirement: each triad is "colored" with one "yes" and two "no's". Certainly, Bell-KS theorem can be viewed as a proof for the contradictions between noncontextual LHV models and quantum mechanics.

However, there are several points which weakened the Bell-KS theorem: first, noncontextual LHV is just a special case of a more general one discussed in Bell's inequalities for $2 \otimes 2$ Hilbert space; second, the Bell-KS theorem does not depend on the entanglement of the state, a spin-1 particle's state is enough, while the Bell's inequalities is not violated by a direct product state[9]. In other words, the contradiction of LHV models and quantum mechanics should depend on the fact that the system is entangled or not; and lastly, there are recent works [10-13] pointed out that finite precision measurement will nullify the Bell-KS theorem, this makes the Bell-KS theorem untestable since the fact that the measurements yet known have finite precision. Comparing with studying of the spin-half particles, the situation of the higher spin case is rather unsatisfying. Recently, M.Zukowski *et al.* [14] showed that higher-dimensional two-particle entanglements are realizable via multipart

beam splitters, and the results presented in their paper move the discussion on entangled higher-than-1/2 spin systems from the realm of gedanken-experiments to real experiments. Being motivated by their work, here we shall let the $3 \otimes 3$ case to be considered in a way similar to what Bell's theorem has done for the $2 \otimes 2$ case. There has been a series of works studying the case that the Hilbert dimension N for each particle is more than two. First results, in 1980-1982, suggested that the conflict between local realism and quantum mechanics diminishes with growing N [15-17]. In the early 1990's Peres and Gisin [18-19] considered certain dichotomic observables applied to maximally entangled pairs of particles, they showed that the violation of local realism survives, while N is growing, but never exceeds the factor $\sqrt{2}$. Recently, D.Kaszlikowski *et al* [20] investigated the general case of two entangled quantum systems defined in N -dimensional Hilbert spaces(they called it "quNits"), and via a numerical linear optimization method they showed that violations of local realism are stronger for two maximally entangled quNits ($3 \leq N \leq 9$) than that for two quNits and they are increase with N , while the two quNit systems is described by a special mixed state. In present paper, only the case $N=3$ is concerned: first we shall derive a inequality for a $3 \otimes 3$ Hilbert space from locality and reality; then, using the singlet state of two spin-1 particles system, we shall show that the inequality is violated. The inequality is satisfied for the direct product state of the two spin-1 particles system.

The singlet state for two spin-1 particles are

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|m_1 = 1\rangle|m_2 = -1\rangle - |m_1 = 0\rangle|m_2 = 0\rangle + |m_1 = -1\rangle|m_2 = 1\rangle), \quad (1)$$

where $|m_i\rangle$ denotes the eigenvector of spin operator \hat{S} along the direction z , $\hat{S}_i(z)|m_i\rangle = m_i|m_i\rangle$, ($m_i = 1, 0, -1$) for particle $i(i=1,2)$. Let $|m'_i\rangle$ to be the eigenvector of $\hat{S}(\beta_i)$, $\hat{S}(\beta_i)|m'_i\rangle = m'_i|m'_i\rangle$ (for simplicity, we have let the directions in the x - z plane, and each direction is viewed as a rotation β along the y axis), there is a connection between $|m_i\rangle$ and $|m'_i\rangle$

$$|m'_i\rangle = \sum_{j=1}^3 D_{ji}(\beta)|m_j\rangle, \quad (2)$$

and the rotation matrix is

$$D(\beta) = \begin{vmatrix} \frac{1+\cos(\beta)}{2} & \frac{\sin(\beta)}{\sqrt{2}} & \frac{1-\cos(\beta)}{2} \\ \frac{\sin(\beta)}{\sqrt{2}} & \cos(\beta) & -\frac{1-\cos(\beta)}{\sqrt{2}} \\ \frac{1-\cos(\beta)}{2} & \frac{\sin(\beta)}{\sqrt{2}} & \frac{1+\cos(\beta)}{2} \end{vmatrix} \quad (3)$$

With the singlet state(1) as a source, particle 1 propagates along the y axis, while particle 2 in the -y axis. A Stern-Gerlach magnetic analyzer is put in a place where particle 1 will arrive at, and it will give the results for spin projection along a direction β_1 , while a similar analyzer, which locates a distance away from the analyzer for particle 1, is used to measure the spin projection along β_2 for particle 2. Now, the singlet state (1) can be written as

$$\begin{aligned} |\Psi\rangle = \frac{1}{\sqrt{3}} \{ & \sin^2(\frac{\beta_1 - \beta_2}{2})|1\rangle|1\rangle - \frac{1}{\sqrt{2}} \sin(\beta_1 - \beta_2)|1\rangle|0\rangle + \cos^2(\frac{\beta_1 - \beta_2}{2})|1\rangle|-1\rangle \\ & + \frac{1}{\sqrt{2}} \sin(\beta_1 - \beta_2)|0\rangle|1\rangle - \cos(\beta_1 - \beta_2)|0\rangle|0\rangle - \frac{1}{\sqrt{2}} \sin(\beta_1 - \beta_2)|0\rangle|-1\rangle \\ & + \cos^2(\frac{\beta_1 - \beta_2}{2})|-1\rangle|1\rangle + \frac{1}{\sqrt{2}} \sin(\beta_1 - \beta_2)|-1\rangle|0\rangle + \sin^2(\frac{\beta_1 - \beta_2}{2})|-1\rangle|-1\rangle \}. \end{aligned} \quad (4)$$

Defining the joint probability correlation

$$P_{m_1 m_2} = \langle \Psi | m_1 \rangle \langle m_1 | \otimes | m_2 \rangle \langle m_2 | \Psi \rangle, (m_1, m_2 = 1, 0, -1) \quad (5)$$

the singlet state in the form (4) give the following joint probabilities:

$$P_{11}(\beta_1, \beta_2) = \frac{1}{3} \sin^4(\frac{\beta_1 - \beta_2}{2}), P_{00} + P_{0,-1} + P_{-1,0} + P_{-1,-1} = \frac{1}{3} [1 + \sin^4(\frac{\beta_1 - \beta_2}{2})]. \quad (6)$$

We should prove that the correlation (6) can not be interpreted by LHV models.

Assuming that the state(1) can be described by a set of parameters λ , LHV models gives the probabilities $p_m(\beta_1, \lambda)$ and $q_n(\beta_2, \lambda)$ for the two results that the spin projection along β_1 is m for particle 1 and the spin projection along β_2 is n for particle 2 respectively($m, n=1,0,-1$). As a cosequence of the relation

$$|m_i = 1\rangle \langle m_i = 1| + |m_i = 0\rangle \langle m_i = 0| + |m_i = -1\rangle \langle m_i = -1| = I, (i = 1, 2) \quad (7)$$

there is a natural conditions here

$$\begin{aligned} 0 & \leq p_1(\beta_1, \lambda) + p_0(\beta_1, \lambda) + p_{-1}(\beta_1, \lambda) \leq 1, \\ 0 & \leq q_1(\beta_2, \lambda) + q_0(\beta_2, \lambda) + q_{-1}(\beta_2, \lambda) \leq 1. \end{aligned} \quad (8)$$

It should be noted that $p_m(\beta_1, \lambda)$ does not depend on the settings of β_2 , while $q_n(\beta_2, \lambda)$ is not depending on β_1 either. They are required by the locality assumption, which asserts that experiments done on one place have no influence on the results of measurement done on the other place located a distance away if two measurements are performed simultaneously, and the joint probability should be

$$P_{mn}(\beta_1, \beta_2) = \int_{\lambda \in \Lambda} p_m(\beta_1, \lambda) q_n(\beta_2, \lambda) \rho(\lambda) d\lambda, \quad (9)$$

while

$$\int_{\lambda \in \Lambda} \rho(\lambda) d\lambda = 1. \quad (10)$$

In order to derive a inequality, we use the following simple algebraic theorem: giving six real numbers x, x', X, y, y', Y , such that $0 \leq x, x' \leq X, 0 \leq y, y' \leq Y$ one must always have

$$-XY \leq xy - xy' + x'y + x'y' - x'Y - Xy \leq 0 \quad (11)$$

The proof of it has been given by Clauser and Horne [21]. Making the identifications $x = p_1(\beta_1, \lambda), x' = p_1(\beta'_1, \lambda), y = q_1(\beta_2, \lambda), y' = q_1(\beta'_2, \lambda)$, and taking $X=Y=1$ (since the natural conditions), one can obtain

$$\begin{aligned} & p_1(\beta_1, \lambda) q_1(\beta_2, \lambda) - p_1(\beta_1, \lambda) q_1(\beta'_2, \lambda) + p_1(\beta'_1, \lambda) q_1(\beta'_2, \lambda) \\ & + (p_0(\beta'_1, \lambda) + p_{-1}(\beta'_1, \lambda)) (q_0(\beta_2, \lambda) + q_{-1}(\beta_2, \lambda)) \leq 1, \end{aligned} \quad (12)$$

through using the natural conditions(8). Integrating over $\rho(\lambda)$, a inequality for $3 \otimes 3$ Hilbert space can be derived :

$$\begin{aligned} S &= P_{11}(\beta_1, \beta_2) - P_{11}(\beta_1, \beta'_2) + P_{11}(\beta'_1, \beta'_2) \\ &+ P_{00}(\beta'_1, \beta_2) + P_{0-1}(\beta'_1, \beta_2) \\ &+ P_{-10}(\beta'_1, \beta_2) + P_{-1-1}(\beta'_1, \beta_2) \leq 1. \end{aligned} \quad (13)$$

It can be easily shown that this inequality will be violated by the quantum joint probability given in eq.(6). Choosing the following sets of the angles $\beta_1 = 0, \beta'_1 = 2\beta_2, \beta'_2 = 3\beta_2$, and $\beta_2 = 147.7^\circ$, we get a contradiction

$$S = 1.12 \leq 1. \quad (14)$$

Certainly, the above S is less than the factor $4/3$ given in case of the multiport beam splitter[14].

If the state is a direct product state, for example

$$|\Psi'\rangle = |1\rangle|0\rangle, \quad (15)$$

now the probability is

$$\begin{aligned} P_{11}(\beta_1, \beta_2) &= \cos^4 \frac{\beta_1}{2} \sin^4 \frac{\beta_2}{2}, P_{00} = \frac{1}{4} \sin^2 \beta_1 \sin^2 \beta_2, P_{0,-1} = \frac{1}{2} \sin^2 \beta_1 \cos^4 \frac{\beta_2}{2}, \\ P_{-1,0} &= \frac{1}{2} \cos^4 \frac{\beta_1}{2} \sin^2 \beta_2, P_{-1,-1} = \sin^4 \frac{\beta_1}{2} \cos^4 \frac{\beta_2}{2}, \end{aligned} \quad (16)$$

then, we have the following form

$$S = \cos^4 \frac{\beta_1}{2} \sin^4 \frac{\beta_2}{2} - \cos^4 \frac{\beta_1}{2} \sin^4 \frac{\beta_2'}{2} + \cos^4 \frac{\beta_1'}{2} \sin^4 \frac{\beta_2'}{2} + (1 - \cos^4 \frac{\beta_1'}{2})(1 - \sin^4 \frac{\beta_2}{2}), \quad (17)$$

while the inequality (11) can be transfered into a form

$$xy - xy' + x'y' + (1 - x')(1 - y) \leq 1. \quad (18)$$

Comparing the above two forms, we know that $S \leq 1$ is always satisfied for direct product state and has no relation to the directions chosen.

In conclusion, for the $3 \otimes 3$ Hilbert space, we have derived a inequality as Bell's inequality for the $2 \otimes 2$ case. For the singlet state for two spin-1 particles system, we show that the inequality will be violated while it is valid for the case when the two particle in a direct product state.

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